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On generalization of optimality conditions for multiobjective semi-infinite problems

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Abstract

In this paper, we study the multiobjective semi-infinite programming problem with inequality constraints, in which the objective and the constraint functions are not necessarily continuous. If Ω is a local cone approximation, we consider the notion of Ω-subdifferential for functions. Then, we present the Karush-Kuhn-Tucker type necessary and sufficient optimality conditions under an Abadie type qualification for the considered problems via Ω-subdifferential.

Keywords: semi-infinite problem, multiobjective optimization, optimality condition, local cone approximation
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1 Introduction

A multiobjective semi-infinite programming (MSIP, in brief) is an optimization problem where two or more objectives are to be minimized on a set of feasible solutions described by arbitrarily many constraint functions. Since this type of problem arises in many engineering problems (e.g., robotics, mathematical physics, optimal control, transportation problems, Chebyshev approximation, etc) became an active field of research in applied mathematics; see [6, 10, 23].

In this paper, we shall use the concept of local cone approximation and the associated subdifferential in order to construct Karush-Kuhn-Tucker (KKT, in short) type necessary and sufficient optimality conditions for the following multiobjective semi-infinite programming problems:

$$(P): \qquad \inf \vartheta(x) := \big(\vartheta_1(x), \dots, \vartheta_p(x)\big) \quad \text{s.t.} \quad \phi_j(x) \le 0, \quad j \in J,$$

where the nonempty index set J is arbitrary (not necessarily finite or equipped with some topology), and the objective functions ϑ_i as $i \in I := \{1, \dots, p\}$ and the constraint functions ϕ_j as $j \in J$ are defined from $\mathbb{R}^n$ to $\mathbb{R}$ (not necessarily differentiable or locally Lipschitz or convex or continuous). When J is finite, (P) coincides with the classic multiobjective optimization problem (MOP, in brief). The KKT optimality conditions are definitely among the most important results in MOP theory; see the books [3, 4] and their references. When $p = 1$, MSIP reduces to the semi-infinite programming problem (SIP, in brief). Optimality conditions for SIP with continuous functions have been studied by many authors; see for example [6, 23] in linear case, [6, 25] in convex case, [21] in quasiconvex case, [13] in DC (difference of convex functions) case, [10] in smooth case, and [12, 18, 14, 17, 26] in locally Lipshitz case. The only

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paper that presents some optimality conditions for SIP with arbitrary functions (continuous or noncontinuous) has been written by Kanzi [15]. The results of [15] are stated via Ω-subdifferential where Ω is a local cone approximation.

There are several works available dealing with optimality conditions for MSIP. For instance, for differentiable MOSIPs, some optimality conditions in Fritz-John (FJ, briefly) type have been presented by Caristi *et al.* [1]. Kanzi and his coauthors present some KKT optimality conditions for MSIP with linear, convex, and locally Lipschitz functions in [5], [20] and [2, 9, 16], respectively. Thus, it should be useful and interesting to study optimality conditions for MSIP with an arbitrary function (continuous or noncontinuous). As the best as our knowledge goes, no paper presents optimality conditions for MSIP via Ω-subdifferential where Ω is a local cone approximation. As the extension of Kanzi's results in [15] to multiobjective SIP, this paper fulls this gap.

The rest of the paper is organized as follows. In Section 2, we present basic definitions as well as some preliminary results which are broadly employed in the paper. In Section 3, we present our main results, and these new results are compared with earlier works in Section 4.

2 Notations and Preliminaries

Given a nonempty set $S \subseteq \mathbb{R}^n$, we denote by $\overline{S}$, int$S$, $conv(S)$, and $cone(S)$ the closure of S, the interior of S, the convex hull of S, and the convex cone (containing the origin) of S, respectively. The polar cone of S is defined by

$$S^{\leq} := \{d \in \mathbb{R}^n \mid \langle x, d\rangle \leq 0, \quad \forall x \in S\},$$

where $\langle ., .\rangle$ exhibits the standard inner product in $\mathbb{R}^n$. Notice that $S^{\leq}$ is always a closed convex cone. The bipolar Theorem states that $\left(S^{\leq}\right)^{\leq} = \overline{cone}(S)$, where $\overline{cone}(S)$ denotes the closed convex cone of S (see [11]).

Theorem 2.1. [11] For a given $S \subseteq \mathbb{R}^n$,

- if S is finite, then $cone(S)$ is closed.
- if S is compact and $0_n \notin conv(S)$, then $cone(S)$ is closed.

Theorem 2.2. [11] If the convex function $\Theta : \mathbb{R}^n \to \mathbb{R}$ attaints its minimum of a convex set $C \subseteq \mathbb{R}^n$ at $x_0 \in C$, then

$$0_n \in \partial\Theta(x_0) + N(C, x_0),$$

where 0_n shows the zero vector in $\mathbb{R}^n$, $N(C, x_0)$ denotes the normal cone on C at x_0, defined as

$$N(C, x_0) := \big\{y \in \mathbb{R}^n \mid \langle y, x - x_0\rangle \leq 0, \qquad \forall x \in C\big\},$$

and $\partial\Theta(x_0)$ denotes the convex subdifferential of Θ at x_0, i.e.,

$$\partial\Theta(x_0) := \{\xi \in \mathbb{R}^n \mid \Theta(x) - \Theta(x_0) \geq \big\langle \xi, x - x_0\big\rangle, \quad \forall x \in \mathbb{R}^n\}.$$

As a consequence of bipolar Theorem, we recall from [11, pp. 137] that if $C \subseteq \mathbb{R}^n$ is an arbitrary set, then

$$N(C^{\leq}, 0_n) = \overline{cone}(C). \tag{2.1}$$

It should be mentioned [11] that if $\Pi := \{C_\gamma \mid \gamma \in \Gamma\}$ is a collection of convex sets in $\mathbb{R}^n$, then:

$$cone\big(\bigcup_{\gamma\in\Gamma} C_\gamma\big) = \bigcup_{\{C_{\gamma_1},\ldots,C_{\gamma_n}\}\subseteq\Pi} \; \bigcup_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n_+} \sum_{\nu=1}^{n} \lambda_\nu C_{\gamma_\nu}, \tag{2.2}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Definition 2.3. A set-valued mapping $\Omega : 2^{\mathbb{R}^n} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is called a local cone approximation, if to each set $S \subseteq \mathbb{R}^n$ and to each point $x_0 \in \mathbb{R}^n$ a cone $\Omega(S, x_0)$ is associated, with the following properties:

- $\Omega(S - x_0, 0_n) = \Omega(S, x_0)$.
- $\Omega\big(T(S), T(x_0)\big) = T\big(\Omega(S, x_0)\big)$, for each non-singular linear mapping $T : \mathbb{R}^n \to \mathbb{R}^n$.
- $\Omega(S \cap U^{x_0}, x_0) = \Omega(S, x_0)$, for each neighborhood U^{x_0} of x_0.
- $\Omega(S, x_0) = \emptyset$, for all $x_0 \notin \overline{S}$.
- $\Omega(S, x_0) = \mathbb{R}^n$, for all $x_0 \in \text{int}(S)$.
- $\Omega(S, x_0) + C \subseteq \Omega(S, x_0)$, for each cone $C \subseteq \mathbb{R}^n$ with $S + C \subseteq S$.

Two important examples for local cone approximation are the contingent cone $\mathcal{T}(S,x)$, and the interior directions cone $\mathcal{I}(S,x)$, with are defined, respectively, as follows:

$$\mathcal{T}(S,x_0) := \Big\{z \in \mathbb{R}^n \mid \exists t_k \downarrow 0, \exists z_k \to z, \text{such that } x_0 + t_k z_k \in S \ \ \forall t \in \mathbb{N}\Big\},$$
$$\mathcal{I}(S,x_0) := \Big\{z \in \mathbb{R}^n \mid \exists U^z, \exists \lambda > 0, \forall t \in (0,\lambda), \forall z^* \in U^z \text{ such that } x_0 + tz^* \in S\Big\},$$

where U^z denotes the neighborhood of z.

Definition 2.4. Let Ω be a local cone approximation. If f is a given function from $\mathbb{R}^n$ to $\mathbb{R}$ and $x_0 \in \mathbb{R}^n$, then

- the extended real-valued function $f^{\Omega}(x_0;.) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, defined by
$$f^{\Omega}(x_0;v) := \inf\big\{r \in \mathbb{R} \mid (v,r) \in \Omega\big(\text{epi}f, (x_0, f(x_0))\big)\big\},$$
is called the Ω-directional derivative of f at x_0, where epi$f$ denotes the epigraph of f, i.e.,
$$\text{epi}f := \big\{(x,r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le r\big\}.$$
- the set
$$\partial_{\Omega} f(x_0) := \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \le f^{\Omega}(x_0;v) \quad \forall v \in \mathbb{R}^n\big\},$$
is called the Ω-subdifferential of f at x_0.

Definition 2.5. A local cone approximation Ω is said to be *perfect* if for all $S \subseteq \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, $\Omega(S,x_0)$ is convex and
$$int\big(\Omega(S,x_0)\big) \subseteq \mathcal{I}(S,x_0).$$

Definition 2.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ be given. A perfect local cone approximation Ω is said to be f^{x_0}-perfect if
$$f^{int(\Omega)}(x_0;0_n) = 0.$$

Definition 2.7. A local cone approximation Ω is said to be *additive* if from its $f_1^{x_0}$-perfectness and $f_2^{x_0}$-perfectness we can conclude its $(f_1+f_2)^{x_0}$-perfectness, i.e.,
$$f_i^{int(\Omega)}(x_0;0_n) = 0, \quad i = 1,2, \text{ then } (f_1+f_2)^{int(\Omega)}(x_0;0_n) = 0.$$

An important example for additive local cone approximation is the Clark tangent cone $\mathcal{TC}(S,x)$, defined as follows:
$$\mathcal{TC}(S,x_0) := \Big\{z \in \mathbb{R}^n \mid \forall x_k \xrightarrow{S} x_0, \forall t_k \downarrow 0, \ \exists z_k \to z, \text{ such that } x_k + t_k z_k \in S \quad \forall t \in \mathbb{N}\Big\},$$
where $x_k \xrightarrow{S} x_0$ means that $\{x_k\}$ is a sequence in S converging to x_0. Also, $\mathcal{TC}$ is $g^{\bar{x}}$-perfect for each locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$; see [4] for details. In fact, if we choose $\Omega = \mathcal{TC}$, that the inclusion int$\mathcal{TC}(S,\bar{x}) \subseteq \mathcal{I}(S,\bar{x})$ holds. Moreover, regarding the associated Ω-directional derivatives, for a locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$
$$f^{\mathcal{TC}}(\bar{x};v) = \limsup_{t\to 0^+ \ x\to\bar{x}} \frac{f(x+tv) - f(x)}{t}, \qquad \forall v \in \mathbb{R}^n,$$
$$\partial_{\mathcal{TC}} f(\bar{x}) = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \le f^{\mathcal{TC}}(\bar{x};v), \quad \forall v \in \mathbb{R}^n\big\},$$
and $f^{\text{int}\mathcal{TC}}(\bar{x};0_n) = 0$. In the following Theorems we summarize some important properties of the Ω-directional derivative and the Ω-subdifferential which are widely used in what follows.

Theorem 2.8. [4, Section 4] Suppose that Ω is a f^{x_0}-perfect local cone approximation. Then

(a) $v \to f^{\Omega}(x_0;v)$ is a convex function.

(b) $\partial_{\Omega} f(x_0)$ is a compact set.

(c) $f^{\Omega}(x_0;v) = \max\big\{\langle \xi, v\rangle \mid \xi \in \partial_{\Omega} f(x_0)\big\}$, for all $v \in \mathbb{R}^n$.

(d) $\partial_{int(\Omega)} f(x_0) = \partial_{\Omega} f(x_0) = \partial_{\overline{\Omega}} f(x_0)$.

Theorem 2.9. [15, Lemma 3.1] Suppose that Ω is a f^{x_0}-perfect local cone approximation. Then

$$f^{int(\Omega)}(x_0;v) \geq \limsup_{t\downarrow 0\ u\to v} \frac{f(x_0+tu)-f(x_0)}{t}, \qquad \forall v \in \mathbb{R}^n.$$

Notice that the subadditive formula

$$\partial_{\Omega}\big(f_1+f_2\big)(x_0) \subseteq \partial_{\Omega} f_1(x_0) + \partial_{\Omega} f_2(x_0), \tag{2.3}$$

is not valid for Ω-subdifferential in general.

3 Main Results

At starting point of this section, we present some properties of Ω-directional derivative of functions.

Theorem 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ be given. If Ω is a local cone approximation, for each $\lambda \geq 0$ we have:

- $(\lambda f)^{\Omega}(x_0;v) = \lambda f^{\Omega}(x_0;v)$, for all $v \in \mathbb{R}^n$.
- $\partial_{\Omega}(\lambda f)(x_0) = \lambda \partial_{\Omega} f(x_0)$.

Proof .

- Let $\lambda \geq 0$ be fixed. We define the linear function $T_\lambda : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as $T_\lambda(u,\tau) := (u, \lambda\tau)$, for all $(u,\tau) \in \mathbb{R}^n \times \mathbb{R}$. By second property of Definition 2.3, we have

$$\begin{aligned}
\Omega\Big(\text{epi}(\lambda f), \big(x_0, \lambda f(x_0)\big)\Big) &= \Omega\Big(T_\lambda(\text{epi}f), T_\lambda\big(x_0, f(x_0)\big)\Big) \\
&= T_\lambda\bigg(\Omega\Big(\text{epi}f, \big(x_0, f(x_0)\big)\Big)\bigg) \\
&= \Big\{(u,\lambda\tau) \mid (u,\tau) \in \Omega\Big(\text{epi}f, \big(x_0, f(x_0)\big)\Big)\Big\}.
\end{aligned}$$

From this we imply that $(\lambda f)^{\Omega}(x_0;v) = \lambda f^{\Omega}(x_0;v)$, for all $v \in \mathbb{R}^n$.

- Since the equality is clearly true for $\lambda = 0$, we suppose that $\lambda > 0$. Owing to above equality, we get

$$\begin{aligned}
\partial_{\Omega}(\lambda f)(x_0) &= \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq (\lambda f)^{\Omega}(x_0;v) \quad \forall v \in \mathbb{R}^n\big\} \\
&= \big\{\xi \in \mathbb{R}^n \mid \langle \xi, v\rangle \leq \lambda f^{\Omega}(x_0;v) \quad \forall v \in \mathbb{R}^n\big\} \\
&= \Big\{\xi \in \mathbb{R}^n \mid \Big\langle \frac{\xi}{\lambda}, v\Big\rangle \leq f^{\Omega}(x_0;v) \quad \forall v \in \mathbb{R}^n\Big\} \\
&= \big\{\lambda\xi' \in \mathbb{R}^n \mid \langle \xi', v\rangle \leq f^{\Omega}(x_0;v) \quad \forall v \in \mathbb{R}^n\big\} \\
&= \lambda \partial_{\Omega} f(x_0).
\end{aligned}$$

□

The following Corollary is a direct consequent of Theorem 3.1.

Corollary 3.2. Let $f : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $\lambda > 0$ be given. If Ω is a f^{x_0}-perfect local cone approximation, then Ω is also a $(\lambda f)^{x_0}$-perfect local cone approximation.

Now, we introduce some symbols and recall a definition. The feasible set of problem (P) is denoted by $\mathcal{F}$, i.e.,

$$\mathcal{F} := \big\{x \in \mathbb{R}^n \mid \phi_j(x) \leq 0, \quad j \in J\big\}.$$

For each $\hat{x} \in \mathcal{F}$, with convention $\bigcup_{\emptyset} X_t = \emptyset$, put

$$J(\hat{x}) := \big\{j \in J \mid \phi_j(\hat{x}) = 0\big\}, \quad \text{and} \quad \mathcal{B}_{\Omega}(\hat{x}) := \textstyle\bigcup_{j\in J(\hat{x})} \partial_{\Omega}\phi_j(\hat{x}).$$

Definition 3.3. A feasible point $\hat{x} \in \mathcal{F}$ is called a properly efficient solution to (P) when there exist some scalars $\alpha_i > 0$ as $i \in I$ such that

$$\sum_{i=1}^{p} \alpha_i\vartheta_i(\hat{x}) \leq \sum_{i=1}^{p} \alpha_i\vartheta_i(x), \qquad \forall x \in \mathcal{F}.$$

Definition 3.4. We say that (P) satisfies the Ω-*Abadie qualification* (Ω-AQ, in brief) at $\hat{x} \in \mathcal{F}$ if

$$\big(\mathcal{B}_{\Omega}(\hat{x})\big)^{\leq} \subseteq \mathcal{T}(\mathcal{F}, \hat{x}).$$

Now, we can present our main result.

Theorem 3.5 (Necessary Optimality Condition). Assume that $\hat{x} \in \mathcal{F}$ is a properly efficient solution of (P), and the additive local cone approximation Ω is $\vartheta_i^{\hat{x}}$-perfect as $i \in I$. If the Ω-AQ is satisfied at $\hat{x}$, then there exist some positive scalars $\alpha_i > 0$ (for $i \in I$) such that

$$0_n \in \partial_{\Omega}\Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)(\hat{x}) + \overline{cone}\big(\mathcal{B}_{\Omega}(\hat{x})\big). \tag{3.1}$$

If in addition, $cone\big(\mathcal{B}_{\Omega}(\hat{x})\big)$ is a closed cone, then there exist some non-negative numbers $\beta_j \geq 0$ ($j \in J(\hat{x})$), with finitely many of them being nonzero, such that

$$0_n \in \partial_{\Omega}\Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j\partial_{\Omega}\phi_j(\hat{x}). \tag{3.2}$$

If in addition, the subadditive formula (2.3) holds for Ω, we have

$$0_n \in \sum_{i=1}^{p} \alpha_i\partial_{\Omega}\vartheta_i(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j\partial_{\Omega}\phi_j(\hat{x}). \tag{3.3}$$

Proof . Since $\hat{x}$ is a properly efficient solution of (P), we can find some positive scalars $\lambda_i > 0$ as $i \in I$ such that

$$\sum_{i=1}^{p} \alpha_i\vartheta_i(\hat{x}) \leq \sum_{i=1}^{p} \alpha_i\vartheta_i x), \qquad \forall x \in \mathcal{F},$$

and hence, $\hat{x}$ is a minimizer of function $\sum\limits_{i=1}^{p} \alpha_i\vartheta_i$ on $\mathcal{F}$. Let $v^* \in \mathcal{T}(\mathcal{F}, \hat{x})$ be given. Then, there exists a sequence $\{v_k\} \subseteq \mathbb{R}^n$ converging to v^* and a positive sequence $\{t_k\} \subseteq \mathbb{R}_+$ converging to zero such that $\hat{x} + t_kv_k \in \mathcal{F}$ for each $k \in \mathbb{N}$. By virtue of Theorem 2.9 we get

$$\begin{aligned}
\Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)^{\mathrm{int}(\Omega)}(\hat{x}; v^*) \;&\geq\; \limsup_{t\downarrow 0\ u\to v^*} \frac{\Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)(\hat{x}+tu) - \Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)(\hat{x})}{t} \\
&\geq\; \limsup_{k\to\infty} \frac{\Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)(\hat{x}+t_kv_k) - \Big(\sum_{i=1}^{p} \alpha_i\vartheta_i\Big)(\hat{x})}{t_k} \\
&\geq\; 0,
\end{aligned} \tag{3.4}$$

where the final inequality holds by local minimality of $\hat{x}$ for $\sum_{i=1}^{p}\alpha_i\vartheta_i$ on $\mathcal{F}$. Let $v \in \big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq}$. Since $\big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq} = \big(\mathcal{B}_\Omega(\hat{x})\big)^{\leq}$ and the Ω-AQ holds, we have $v \in \mathcal{T}(\mathcal{F},\hat{x})$, and hence $\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};v) \geq 0$ by (3.4). Thus,

$$\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};v) \geq 0, \qquad \forall v \in \big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq}. \tag{3.5}$$

Because $\vartheta_i^{\text{int}(\Omega)}(\hat{x};0_n) = 0$, we have $(\alpha_i\vartheta_i)^{\text{int}(\Omega)}(\hat{x};0_n) = 0$ for $i \in I$ by Theorem 3.1. From this and additivity assumption of Ω, we get $\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};0_n) = 0$. This equality, (3.5), and the fact that $0_n \in \big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq}$ (by Definition of polar cone) imply that the following optimization problem has a local solution at $\widetilde{v} := 0_n$:

$$\begin{aligned} &\min\ \Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};v) \\ &\text{s.t.}\ \ v \in \big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq}. \end{aligned}$$

Since the objective function and the constraint set of the above problem are convex (see Theorem 2.8(a)), by Theorem 2.2 we give

$$0_n \in \partial\Big(\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};\cdot)\Big)(0_n) + N\Big(\big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq}, 0_n\Big). \tag{3.6}$$

At this point, owing to (2.1), we get

$$N\Big(\big(\overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big)\big)^{\leq}, 0_n\Big) = \overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big), \tag{3.7}$$

and with regard to the Definition of ∂_Ω, and $\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};0_n) = 0$, we have

$$\begin{aligned} \partial\Big(\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};\cdot)\Big)(0_n) =& \Big\{\xi \in \mathbb{R}^n \mid \Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};w) - \Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};0_n) \geq \langle \xi, w - 0_n\rangle,\ \forall\, w \in \mathbb{R}^n\Big\} \\ =& \Big\{\xi \in \mathbb{R}^n \mid \Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)^{\text{int}(\Omega)}(\hat{x};w) \geq \langle \xi, w\rangle,\ \forall\, w \in \mathbb{R}^n\Big\} \\ =& \partial_{\text{int}(\Omega)}\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)(\hat{x}). \end{aligned}$$

The above equality, (3.6) and (3.7), imply that

$$0_n \in \partial_{\text{int}(\Omega)}\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)(\hat{x}) + \overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big) = \partial_\Omega\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)(\hat{x}) + \overline{cone}\big(\mathcal{B}_\Omega(\hat{x})\big),$$

where the final equality holds by Theorem 2.8(d). Thus, (3.1) is proved. Consequently, if $cone\big(\mathcal{B}_\Omega(\hat{x})\big)$ is closed, then

$$0_n \in \partial_\Omega\Big(\sum_{i=1}^{p}\alpha_i\vartheta_i\Big)(\hat{x}) + cone\big(\mathcal{B}_\Omega(\hat{x})\big). \tag{3.8}$$

The convexity of $\partial_\Omega\phi_j(\hat{x})$ as $j \in J(\hat{x})$ and (2.2) concludes that we can find some $\beta_j \geq 0$ as $j \in J(\hat{x})$, with finitely many of them being nonzero, such that

$$cone\big(\mathcal{B}_\Omega(\hat{x})\big) = \sum_{j \in J(\hat{x})} \beta_j\partial_\Omega\phi_j(\hat{x}).$$

The above equality and (3.8) imply

$$0_n \in \partial_\Omega\Big(\sum_{i=1}^{p} \alpha_i \vartheta_i\Big)(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j \partial_\Omega \phi_j(\hat{x}),$$

and hence, (3.2) is proved. If the subadditive formula (2.3) holds for Ω, we have

$$\partial_\Omega\Big(\sum_{i=1}^{p} \alpha_i \vartheta_i\Big)(\hat{x}) \subseteq \sum_{i=1}^{p} \alpha_i \partial_\Omega \vartheta_i(\hat{x}),$$

and hence

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_\Omega \vartheta_i(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j \partial_\Omega \phi_j(\hat{x}).$$

Therefore, (3.3) is proved, and the proof is complete. □

For presenting the sufficient optimality condition, the following theorem and definition will be required.

Theorem 3.6. Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ be given. If the subadditive formula (2.3) holds for $f_k^{x_0}$-perfect local cone approximation Ω as $k = 1, 2$, we have

$$\big(f_1 + f_2\big)^\Omega(x_0; v) \leq f_1^\Omega(x_0; v) + f_2^\Omega(x_0; v), \qquad \forall v \in \mathbb{R}^n.$$

Proof . Owing to Theorem 2.8[c], for each $v \in \mathbb{R}^n$ we have

$$\begin{array}{rcl} \big(f_1 + f_2\big)^\Omega(x_0; v) & = & \max\big\{\langle \xi, v\rangle \mid \xi \in \partial_\Omega\big(f_1 + f_2\big)(x_0)\big\} \\ & \leq & \max\big\{\langle \xi, v\rangle \mid \xi \in \partial_\Omega f_1(x_0) + \partial_\Omega f_2(x_0)\big\} \\ & = & \max\big\{\langle \xi_1 + \xi_2, v\rangle \mid \xi_1 \in \partial_\Omega f_1(x_0),\ \xi_2 \in \partial_\Omega f_2(x_0)\big\} \\ & = & \max\Big(\big\{\langle \xi_1, v\rangle \mid \xi_1 \in \partial_\Omega f_1(x_0)\big\} + \big\{\langle \xi_2, v\rangle \mid \xi_2 \in \partial_\Omega f_2(x_0)\big\}\Big) \\ & \leq & \max\big\{\langle \xi_1, v\rangle \mid \xi_1 \in \partial_\Omega f_1(x_0)\big\} + \max\big\{\langle \xi_2, v\rangle \mid \xi_2 \in \partial_\Omega f_2(x_0)\big\} \\ & = & f_1^\Omega(x_0; v) + f_2^\Omega(x_0; v). \end{array}$$

The proof is complete. □

Definition 3.7. Let Ω be a local cone approximation and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function. The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Ω_η- invex at $x_0 \in \mathbb{R}^n$ when

$$f(x) - f(x_0) \geq f^\Omega\big(x_0; \eta(x, x_0)\big), \qquad \forall x \in \mathbb{R}^n.$$

Note that the above Definition is a generalization of η-invex functions, introduced in [7, 8, 24]. The following simple Corollary will be used in sequel.

Corollary 3.8. Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ and nonnegative scalars λ_1, λ_2 be given. If the subadditive formula (2.3) holds for $f_k^{x_0}$-perfect local cone approximation Ω and f_k is Ω_η- invex at x_0 for $k = 1, 2$, then $\lambda_1 f_1 + \lambda_2 f_2$ is Ω_η-invex at x_0.

Proof . Since for each $x \in \mathbb{R}^n$, we have

$$f_k(x) - f_k(x_0) \geq f_k^\Omega\big(x_0; \eta(x, x_0)\big), \qquad k = 1, 2,$$

then

$$\lambda_1\big(f_1(x) - f_1(x_0)\big) + \lambda_2\big(f_2(x) - f_2(x_0)\big) \geq \lambda_1 f_1^\Omega\big(x_0; \eta(x, x_0)\big) + \lambda_2 f_2^\Omega\big(x_0; \eta(x, x_0)\big).$$

Owing to the Theorems (3.1) and (3.6), the above inequality implies that

$$\big(\lambda_1 f_1 + \lambda_2 + f_2\big)(x) - \big(\lambda_1 f_1 + \lambda_2 + f_2\big)(x_0) \geq \big(\lambda_1 f_1 + \lambda_2 + f_2\big)^\Omega\big(x_0; \eta(x, x_0)\big),$$

and the result is proved. □

Now, the sufficient optimality condition can be stated as follows.

Theorem 3.9 (Sufficient Optimality Condition). Suppose that the subadditive formula (2.3) holds for $\vartheta_i^{\hat{x}}$-perfect local cone approximation Ω and ϑ_i is Ω_η- invex at $\hat{x} \in \mathcal{F}$ for $i \in I$. Furthermore, assume that there exist a finite index set $\hat{J} \subseteq J(\hat{x})$, scalars $\alpha_i > 0$ for $i \in I$, and numbers $\beta \geq 0$ for $j \in \hat{J}$ such that

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_\Omega \vartheta_i(\hat{x}) + \sum_{j \in \hat{J}} \beta_j \partial_\Omega \phi_j(\hat{x}).$$

If the ϕ_j (for $j \in \hat{J}$) functions are Ω_η- invex at $\hat{x}$, then $\hat{x}$ is a properly efficient solution for (P).

Proof . According to the assumption, we can find some $\hat{\xi}_i \in \partial_\Omega \vartheta_i(\hat{x})$ and $\hat{\zeta}_j \in \partial_\Omega \phi_j(\hat{x})$ as $(i,j) \in I \times \hat{J}$ such that

$$\sum_{i=1}^{p} \alpha_i \hat{\xi}_i + \sum_{j \in \hat{J}} \beta_j \hat{\zeta}_j = 0_n. \tag{3.9}$$

We claim that

$$\sum_{i=1}^{p} \alpha_i \vartheta_i(\hat{x}) \leq \sum_{i=1}^{p} \alpha_i \vartheta_i(x), \qquad \forall x \in \mathcal{F}. \tag{3.10}$$

Suppose on the contrary, there exists $x^* \in \mathcal{F}$ such that $\sum_{i=1}^{p} \alpha_i \vartheta_i(\hat{x}) > \sum_{i=1}^{p} \alpha_i \vartheta_i(x^*)$. This inequality, the Ω_η- invexity of ϑ_i as $i \in I$, the validity of subadditive formula (2.3), Theorem 2.8, and Corollary (3.8) imply that

$$\begin{aligned}
\Big\langle \sum_{i=1}^{p} \alpha_i \hat{\xi}_i, \eta(x^*, \hat{x}) \Big\rangle &\leq \max \Big\{ \Big\langle \sum_{i=1}^{p} \alpha_i \xi_i, \eta(x^*, \hat{x}) \Big\rangle \mid \sum_{i=1}^{p} \alpha_i \xi_i \in \partial_\Omega \Big(\sum_{i=1}^{p} \alpha_i \vartheta_i \Big)(\hat{x}) \Big\} \\
&\leq \max \Big\{ \sum_{i=1}^{p} \alpha_i \big\langle \xi_i, \eta(x^*, \hat{x}) \big\rangle \mid \xi_i \in \partial_\Omega \vartheta_i(\hat{x}) \Big\} \\
&\leq \sum_{i=1}^{p} \alpha_i \max \big\{ \big\langle \xi_i, \eta(x^*, \hat{x}) \big\rangle \mid \xi_i \in \partial_\Omega \vartheta_i(\hat{x}) \big\} \\
&= \sum_{i=1}^{p} \alpha_i \vartheta_i^{\Omega}\big(\hat{x}, \eta(x^*, \hat{x})\big) = \Big(\sum_{i=1}^{p} \alpha_i \vartheta_i \Big)^{\Omega} \big(\hat{x}, \eta(x^*, \hat{x})\big) \\
&\leq \Big(\sum_{i=1}^{p} \alpha_i \vartheta_i \Big)(x^*) - \Big(\sum_{i=1}^{p} \alpha_i \vartheta_i \Big)(\hat{x}) < 0.
\end{aligned} \tag{3.11}$$

On the other hand, multiplying (3.9) by $\eta(x^*, \hat{x})$, we get

$$\Big\langle \sum_{i=1}^{p} \alpha_i \hat{\xi}_i, \eta(x^*, \hat{x}) \Big\rangle + \Big\langle \sum_{j \in \hat{J}} \beta_j \hat{\zeta}_j, \eta(x^*, \hat{x}) \Big\rangle = 0.$$

This equality and virtue of (3.11) conclude that $\Big\langle \sum_{j \in \hat{J}} \beta_j \hat{\zeta}_j, \eta(x^*, \hat{x}) \Big\rangle > 0$, and hence, $\big\langle \hat{\zeta}_{j_0}, \eta(x^*, \hat{x}) \big\rangle > 0$ for some $j_0 \in \hat{J}$ with $\beta_{j_0} > 0$. Now, Ω_η- invexity of ϕ_{j_0} and $j_0 \in \hat{J} \subseteq J(\hat{x})$ imply that

$$\phi_{j_0}(x^*) = \phi_{j_0}(x^*) - \phi_{j_0}(\hat{x}) \geq \big\langle \hat{\zeta}_{j_0}, \eta(x^*, \hat{x}) \big\rangle > 0.$$

This is contradiction, since it was assumed that $x^* \in \mathcal{F}$. This contradiction proves the claim (3.10), and the result is proved. $\Box$

4 Comparison with Earlier Works

In this section we compare our developments with some known results. We suppose that the functions ϑ_i and ϕ_j as $(i,j) \in I \times J$ are locally Lipschitz. Since $\mathcal{TC}$ is a additive local cone approximation which is $\vartheta_i^{\hat{x}}$-perfect for all $i \in I$, and subadditive formula (2.3) holds, the following Theorem, which is proved in [12], is a direct consequence of Theorem 3.5.

Theorem 4.1. Assume that $\hat{x} \in F$ is a properly efficient solution of (P) with locally Lipschitz functions. If the $\mathcal{TC}$-AQ is satisfied at $\hat{x}$, then there exist some positive scalars $\alpha_i > 0$ (for $i \in I$) such that

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_{\mathcal{TC}} \vartheta_i(\hat{x}) + \overline{cone}\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big).$$

If in addition, $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$ is a closed cone, then there exist some non-negative numbers $\beta_j \geq 0$ $(j \in J(\hat{x}))$, with finitely many of them being nonzero, such that

$$0_n \in \sum_{i=1}^{p} \alpha_i \partial_{\mathcal{TC}} \vartheta_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial_{\mathcal{TC}} \phi_j(\hat{x}). \tag{4.1}$$

Note that the above Theorem has been proven in [9] under a qualification condition that is stronger than $\mathcal{TC}$-AQ. So, a part of [9] is an Corollary of Theorem 4.1. Also, if we consider an arbitrary index set T and add the equality constraints $\psi_t(x) = 0$ as $t \in T$ to (P) for locally Lipschitz functions $\psi : \mathbb{R}^n \to \mathbb{R}$, by rewriting each $\psi_t(x) = 0$ as $\psi_t(x) \leq 0$ and $-\psi_t(x) \leq 0$, we find the necessary optimality conditions that are presented in [19]. Moreover, taking $p = 1$ in (P), we obtain the results of [14] from Theorem 4.1. Observe that, the sufficient conditions that are stated in [12], are consequences of Theorem 3.9. We should mention that, by tacking the local cone approximation Ω, Theorems 3.5 and 3.9 imply the necessary and sufficient optimality conditions under Meachel-Penot, Dini-Hadamard, and Clarke-Rockafallar subdifferentials; see [4] for study the Definitions and properties. Also, if the functions ϑ_i and ϕ_j are convex for $i \in I$ and $j \in J$, Theorems 3.5 and 3.9 conclude the results in [5]. As a very special case, we can deduce the results of [20] for linear multiobjective semi-infinite optimization problem, from Theorems 3.5 and 3.9. The following example shows that the condition of closedness of $cone\big(\mathcal{B}_{\Omega}(\hat{x})\big)$ can not be waved for getting (4.1) in Theorem 4.1.

Example 4.2. Suppose that $\vartheta_1(x) = \vartheta_2(x) = -x_1$, $n := 2$, $J := \mathbb{N} \cup \{0\}$, and $\phi_j(x)$ is the support function of the following set

$$U_j = \{x \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1 - j)^2 \leq (1+j)^2,\ x_1 \geq 0,\ x_2 \geq 0\}.$$

The set of feasible solutions for the problem (P) is

$$S = \{x \in \mathbb{R}^2 \mid x_1 \leq 0,\ x_2 \leq 0\ \}.$$

It is easy to verify that ϑ_i and ϕ_j as $i \in I$ and $j \in J$ are locally Lipschitz functions and $\hat{x} = 0_2$ is a properly efficient solution for (P). We observe that

$$\mathcal{TC}(S, \hat{x}) = S, \qquad \partial_{\mathcal{TC}} \phi_j(\hat{x}) = U_j, \qquad \partial_{\mathcal{TC}} \vartheta_1(\hat{x}) = \partial_{\mathcal{TC}} \vartheta_2(\hat{x}) = \{(-1, 0)\},$$
$$cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big) = \{x \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 < 0\} \cup \{0_2\}.$$

Since $\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)^{\leq} = S$, $\mathcal{TC}$-AQ is satisfied at $\hat{x}$. Note that $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$ is not closed. It is easy to see that there is no sequence of scalars as in Theorem 4.1 satisfying (4.1). Moreover, a short calculation shows that for $\alpha_1 = \alpha_2 = \dfrac{1}{2}$ we have

$$0_2 \in \alpha_1 \partial_{\mathcal{TC}} \vartheta_1(\hat{x}) + \alpha_2 \partial_{\mathcal{TC}} \vartheta_2(\hat{x}) + \overline{cone}\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big).$$

As the final point, we observe that the restrictive assumption in Theorem 4.1 is the closedness of $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$. Let us mention some important conditions that ensure the closedness of $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$.

(i): If J is a finite set and ϕ_j functions are continuously differentiable as $j \in J$, their Clarke subdifferentials contain single element, and so, the closedness condition of $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$ automatically holds by Theorem 2.1.

(ii): Whenever J is a finite set and the functions ϕ_j are piecewise affine, their Clarke subdifferentials are (unions of) points and polyhedrons, and hence, $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$ is finitely generated and naturally closed.

(iii): According to compactness of Clarke subdifferential, and using Theorem 2.1, we conclude that if $0_n \notin conv\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$, then $cone\big(\mathcal{B}_{\mathcal{TC}}(\hat{x})\big)$ is closed.

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