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Reproducing kernel method to solve a class of variable delay integro-differential equations

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Abstract

This paper deals with an efficient method to solve Variable Delay Integro-Differential Equations (VDIDEs). We use a different implementation of the general form of the Reproducing kernel Method (RKM). We seek to apply RKM without using the orthogonalization process. The main purpose of this technique for the VDIDE is to implement it in large intervals so that an appropriate approximate solution can be obtained, and also a valid error analysis can be provided. This method significantly increases the accuracy of approximate solutions in small intervals. The accuracy of theoretical results is also illustrated by solving two numerical examples.

Keywords: Integro-Differential Equations, Delay differential equation, Initial value problem, Reproducing kernel method, Error analysis
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1 Introduction

This paper is concerned with an efficient semi-analytical method to solve VDIDE as follows,

$$\begin{cases} u'(x)+a(x)u(x)+b(x)u(x-\tau(x))+\int_{\lambda(x)}^{x} c(s,x)u(s)ds=f(x), & x\geq 0,\\ u(x)=h(x), \quad -\tau(x)\leq x<0, \quad \tau(x)\geq 0, & \end{cases} \tag{1.1}$$

where $\lambda(x)=\max\{x-\tau(x),0\}$ and $\tau(x)$ is a delayed function and $a(x), b(x), f(x)$ are continuous functions, the initial function $h(x)$ is continuous on $[-\tau(x),0]$. The existence and uniqueness of the solution and stability analysis for VDIDE have been studied in [34, 33, 10, 15, 19, 16, 17, 20, 12]. In [11], M. Cui and Y. Lin solved Eq. (1.1) using the general form of RKM. VDIDEs have been applied in several real-world problems such as biological population, grazing systems, wave propagation, nuclear reactors, viscoelasticity and large-scale systems [19]. To this end, many authors have proposed various computational methods to solve VDIDEs [6, 9, 26, 30, 29, 31, 32, 8], and some important delay problems are discussed in [13, 21, 2, 18, 22]. In this research we use a different implementation of general form of RKM that is presented in [28, 27] by Wang et al. and this idea is denoted as RKM without using the orthogonalization process that is completely introduced in [23, 24, 25, 1, 3].

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We define the reproducing kernel space $W^2[0,\infty]$,

$$\begin{aligned}
&W^2[0,\infty] = \Big\{u(x)\big|u'(x) \text{ is absolutely continuous, } u'(x) \in L^2[0,\infty],\ u(0)=0\Big\},\\
&< u_1(x), u_2(x) >_{W^2} = \int_0^\infty 4u_1(x)u_2(x) + 5{u_1}'(x){u_2}'(x) + {u_1}''(x){u_2}''(x)dx,\\
&\|u(x)\|_{W^2} = \sqrt{< u,u >_{W^2}}, \qquad u_1(x), u_2(x) \in W^2,
\end{aligned} \tag{1.2}$$

and $W^1[0,\infty]$,

$$\begin{aligned}
&W^1[0,\infty] = \Big\{u(x)\big|u(x) \text{ is absolutely continuous, } u'(x) \in L^2[0,\infty]\Big\},\\
&< u_1(x), u_2(x) >_{W^1} = \int_0^\infty u_1(x)u_2(x) + {u_1}'(x){u_2}'(x)dx,\\
&\|u(x)\|_{W^1} = \sqrt{< u,u >_{W^1}}, \qquad u_1(x), u_2(x) \in W^1,
\end{aligned} \tag{1.3}$$

and we consider the reproducing kernel functions for spaces $W^2[0,\infty]$ and $W^1[0,\infty]$ in the following form [28] as well,

$$R_y(x) = \frac{1}{12}e^{-2x-2y} - \frac{e^{-x-y}}{6} + \begin{cases} \frac{e^{x-y}}{6} - \frac{1}{12}e^{2x-2y}, & x \le y,\\ \frac{e^{y-x}}{6} - \frac{1}{12}e^{2y-2x}, & y < x, \end{cases} \tag{1.4}$$

$$r_y(x) = \frac{e^{-x-y}}{2} + \begin{cases} \frac{-e^{x-y}}{2}, & x \le y,\\ \frac{-e^{y-x}}{2}, & y < x. \end{cases} \tag{1.5}$$

If we consider $L(u(x)) \equiv u'(x) + a(x)u(x) + b(x)u(x-\tau(x)) + \int_{\lambda(x)}^x c(s,x)u(s)ds$ on $[0,\mathcal{T}]$ then it is followed by,

$$\begin{aligned}
&L\big(u(x)\big) =\\
&\begin{cases} u'(x) + a(x)u(x) + \displaystyle\int_{\lambda(x)}^x c(s,x)u(s)ds, & 0 \le x < \tau(x),\\ u'(x) + a(x)u(x) + b(x)u(x-\tau(x)) + \displaystyle\int_{\lambda(x)}^x c(s,x)u(s)ds, & x \ge \tau(x), \end{cases}
\end{aligned} \tag{1.6}$$

where $\lambda(x) = max\{x - \tau(x), 0\}$

$$F(x) = \begin{cases} f(x) + b(x)h(x-\tau(x)), & 0 \le x < \tau(x),\\ f(x), & x \ge \tau(x), \end{cases} \tag{1.7}$$

such that $L : W^2[0,\infty] \longrightarrow W^1[0,\infty]$ is a bounded linear operator, and we also suppose $r_y(x)$ is reproducing kernel function for $W^1[0,\infty]$. We also choose a dense set $\{x_i\}_{i=1}^\infty$ on $[0,\mathcal{T}]$ and define, $\varphi_i(x) = r_y(x)|_{y=x_i}$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is adjoint operator of L such that L^{-1} exists, and $\xi_i(x) = R_{x_i}(x)$ are complete function system in $W^2[0,\infty]$, see [23]. Suppose, the following function

$$u(x) = \sum_{i=1}^\infty c_i\xi_i(x), \tag{1.8}$$

is exact solution of Eq. (1.1) where $\{x_i\}_{i=1}^\infty$ are dense points on domain of Eq. (1.1). We consider approximate solution of Eq. (1.1) as,

$$u_n(x) = \sum_{i=1}^n c_i\xi_i(x), \tag{1.9}$$

where n is number of collocation points. For determining the unknown coefficients c_i, we solve following system of algebraic equations

$$\begin{bmatrix} L\xi_1(x_1) & L\xi_2(x_1) & L\xi_3(x_1) & \dots & L\xi_n(x_1)\\ L\xi_1(x_2) & L\xi_2(x_2) & L\xi_3(x_2) & \dots & L\xi_n(x_2)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ L\xi_1(x_{n-1}) & L\xi_2(x_{n-1}) & L\xi_3(x_{n-1}) & \dots & L\xi_n(x_{n-1})\\ L\xi_1(x_n) & L\xi_2(x_n) & L\xi_3(x_n) & \dots & L\xi_n(x_n) \end{bmatrix} \begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_{n-1}\\ c_n \end{bmatrix} = \begin{bmatrix} F(x_1)\\ F(x_2)\\ \vdots\\ F(x_{n-1})\\ F(x_n) \end{bmatrix}. \tag{1.10}$$

2 Convergence Analysis and Error Bound

Now we are going to show that the solution of the system of algebraic equations (1.10) exists and unique.

Remark 2.1. Reproducing kernel $R_y(x)$ is a positive definite kernel, [5]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} R_{x_i}(x_j)m_i m_j \geq 0. \tag{2.1}$$

Remark 2.2. For any $y \in [0, \mathcal{T}]$ which is fixed, reproducing kernel $R_x(x) \geq 0$ and $R_x(x) = 0$ if and only if $W^2 = \{0\}$. See, [5].

Theorem 2.1. For any arbitrary nonzero vector $M = [m_1, m_2, \ldots, m_n]^T$ the matrix $[L\xi_i(x_j)]_{i=1,2,\ldots,n}^{j=1,2,\ldots,n}$ is a positive definite matrix.

Proof . Suffice it to show that,

$$M^T L\xi_i(x_j)M > 0, \tag{2.2}$$

where,

$$\begin{aligned} &\begin{bmatrix} m_1\, m_2\, \ldots\, m_{n-1}\, m_n \end{bmatrix} \begin{bmatrix} L\xi_1(x_1) & L\xi_2(x_1) & L\xi_3(x_1) & \ldots & L\xi_n(x_1) \\ L\xi_1(x_2) & L\xi_2(x_2) & L\xi_3(x_2) & \ldots & L\xi_n(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L\xi_1(x_{n-1}) & L\xi_2(x_{n-1}) & L\xi_3(x_{n-1}) & \ldots & L\xi_n(x_{n-1}) \\ L\xi_1(x_n) & L\xi_2(x_n) & L\xi_3(x_n) & \ldots & L\xi_n(x_n) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{n-1} \\ m_n \end{bmatrix} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} L\xi_i(x_j)m_j m_i = L\sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i(x_j)m_j m_i = L\sum_{i=1}^{n}\sum_{j=1}^{n} R_{x_i}(x_j)m_j m_i. \end{aligned} \tag{2.3}$$

From Remark 2.1 and 2.2, $\sum_{i=1}^{n}\sum_{j=1}^{n} R_{x_i}(x_j)m_j m_i$ is non-negative and since L^{-1} exists, therefore, if we assume that $\sum_{i=1}^{n}\sum_{j=1}^{n} R_{x_i}(x_j)m_j m_i = 0$ then it follows that $M = 0$ and the proof is complete. □

We have following Theorem to show that $u_n(x)$ and its derivative $u'_n(x)$ are uniformly convergent to $u(x)$ and $u'(x)$ respectively.

Theorem 2.2. Let $\mathcal{B} = \{u_n(x) |\ \|u_n(x)\|_{W^2} \leq \theta\}$ where is compact set in $C[0, \mathcal{T}]$ and θ is a constant, then, approximate solution and its derivative are uniformly convergent.

Proof . See [11] □

Theorem 2.3. If the Eq. (1.1) has solution $u(x)$, then the present method is stable in the reproducing kernel space W^2.

Proof . Let $L(u_n(x)) = F_n(x)$ and $F(x) = F_n(x) + \varepsilon_n(x)$, where $\varepsilon_n(x)$ is a perturbation and $\varepsilon_n(x) \xrightarrow{W^2} 0$ $(n \to 0)$. From the equation (1.8), (1.9)

$$u(x) = \sum_{i=1}^{\infty} c_i\xi_i(x), \qquad u_n(x) = \sum_{i=1}^{n} c_i\xi_i(x), ,$$

for $F(x), F_n(x) \in W^1$, we have,

$$L(u(x) - u_n(x)) = F(x) - F_n(x) = \varepsilon_n(x),$$

and therefor

$$u(x) - u_n(x)) = L^{-1}\varepsilon_n(x),$$

from the continuity of L^{-1} and $\varepsilon_n(x) \xrightarrow{W^2} 0$ $(n \to 0)$, we have

$$\lim_{n\to\infty} \|u(x) - u_n(x)\|_{W^2} \leq \|L^{-1}\|\|\varepsilon_n(x)\|_{W^2} = 0.$$

□

Remark 2.3. To provide a valid error bound for the present method, we need the appropriate accuracy of the approximate solution and also its derivative, so the proof of the error analysis theorem is completely related to the present method and the proof of the convergence Theorem 2.2.

Theorem 2.4. Suppose $u_n(x)$ is the approximate solution of the Eq. (1.1) in space W^2 and $u(x)$ is the exact solution. If $u''(x) \in C[0,\mathcal{T}]$ and $\|u_n''(x)\|_\infty$ are bonded then,

$$\|u(x) - u_n(x)\|_\infty \leq ch^2,$$
$$\|u'(x) - u_n'(x)\|_\infty \leq \tilde{c}h,$$

where $c, \tilde{c}$ are positive constants, $\|u(x)-u_n(x)\|_\infty = \max_{x\in[0,\mathcal{T}]} |u(x)-u_n(x)|$, $\|u'(x)-u_n'(x)\|_\infty = \max_{x\in[0,\mathcal{T}]} |u'(x)-u_n'(x)|$ and $h = max_{1\leq i\leq n}|x_{i+1} - x_i|$, and n is number of collocation points on $[0,\mathcal{T}]$.

Proof . According to the strategy in references [4, 14, 7], in each $[x_i, x_{i+1}] \subset [0,\mathcal{T}]$ we can write,

$$u'(x) - u_n'(x) = u'(x) - u'(x_i) + u_n'(x_i) - u_n'(x) + u'(x_i) - u_n'(x_i). \tag{2.4}$$

We write two terms of Taylor series expansion of $u'(x)$ at the point x_i as,

$$u'(x) \approx u'(x_i) + (x - x_i)u''(x_i),$$

since $u''(x) \in C[0,\mathcal{T}y]$ constants c_1 exist such that for all $x \in [0,\mathcal{T}]$, $|u''(x)| \leq c_1$,

$$\|u'(x) - u'(x_i)\|_\infty \leq c_1 h. \tag{2.5}$$

Moreover, we can write

$$u_n'(x_i) - u_n'(x) = -\int_{x_i}^{x} u_n''(s)ds,$$
$$|u_n'(x_i) - u_n'(x)| \leq \int_{x_i}^{x} |u_n''(s)|ds,$$

since $\|u_n''(x)\|_\infty$ are bonded, we have

$$\|u_n'(x_i) - u_n'(x)\|_\infty \leq c_2 h. \tag{2.6}$$

From Theorem 2.2, approximate solution $u_n^m(x)$, that is obtained by the present method are uniformly convergent to exact solution of the Eq. (1.1), $u^m(x)$ where $m = 0, 1$, furthermore for all $\varepsilon_1, \varepsilon_2 > 0$, there exist n_1, n_2 sufficiently large such that,

$$|u'(x_i) - u_n'(x_i)| \leq \epsilon_1, \qquad |u(x_i) - u_n(x_i)| \leq \epsilon_2, \tag{2.7}$$

by combining above equations we have,

$$\|u'(x) - u_n'(x)\|_\infty \leq \tilde{c}h. \tag{2.8}$$

We know have,

$$u(x) - u_n(x) = u(x_i) - u_n(x_i) + \int_{x_i}^{x} (u'(t) - u_n'(t))dt, \tag{2.9}$$

by combining Eqs. (2.7), (2.8) and (2.9) we have,

$$\|u(x) - u_n(x)\|_\infty \leq ch^2.$$

□

3 Numerical Examples

In this section Software package Mathematica 12.1 and absolute errors are used to solve and compare numerical examples results. In Tables 2 and 3, convergence orders for different values of n (number of collocation points) are given. Figures 1, 2 and Table 1 are given to compare the absolute errors with [11] where $E_n = Max_{x\in[0,\mathcal{T}]}|u_n(x) - u(x)|$ and $E_n' = Max_{x\in[0,\mathcal{T}]}|u_n'(x) - u'(x)|$.

Table 1: Max absolute errors for Examples 3.1, 3.2 on $x \in [0, 1]$

Example 3.1			Example 3.2		
PM		[11]	PM		[11]
E_{100}	E'_{100}	E_{200}	E_{100}	E'_{100}	E_{200}
9.30×10^{-6}	1.40×10^{-3}	7.40×10^{-4}	8.50×10^{-5}	1.40×10^{-2}	4.10×10^{-3}

Table 2: Convergence order for Example 3.1 on $x \in [0, 1]$

E_{10}	E_{20}	$Log_2 \frac{E_{10}}{E_{20}}$	E_{40}	$Log_2 \frac{E_{20}}{E_{40}}$	E_{80}	$Log_2 \frac{E_{40}}{E_{80}}$
1.60×10^{-3}	3.40×10^{-4}	2.23447	8.50×10^{-5}	2.00	2.10×10^{-5}	2.01707
E'_{10}	E'_{20}	$Log_2 \frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2 \frac{E'_{20}}{E'_{40}}$	E'_{80}	$Log_2 \frac{E'_{40}}{E'_{80}}$
2.10×10^{-2}	1.00×10^{-2}	1.07039	5.00×10^{-3}	1.00	2.40×10^{-3}	1.05889

Example 3.1. [11]

$$\begin{cases} u'(x) + u(x) + u(x - \frac{1}{5}) + \int_{\lambda(x)}^{x} u(s)ds = f(x), & x \geq 0, \\ u(x) = x, \quad -\frac{1}{5} \leq x < 0. \end{cases}$$

where $u(x) = \frac{x}{1+x^2}$ is exact solution and

$$f(x) = \begin{cases} \frac{2\left(x^3-x^2+x+1\right)+\left(x^2+1\right)^2 ln\left(x^2+1\right)}{2(x^2+1)^2}, & 0 \leq x < \frac{1}{5}, \\ -\frac{2x^2}{(x^2+1)^2} + \frac{x}{x^2+1} + \frac{1}{x^2+1} + \\ \frac{1}{2}\left(ln\left(x^2+1\right) - ln\left(x^2 - \frac{2x}{5} + \frac{26}{25}\right)\right) + \frac{x-\frac{1}{5}}{\left(x-\frac{1}{5}\right)^2+1}, & \frac{1}{5} \leq x. \end{cases}$$

Example 3.2. [11]

$$\begin{cases} u'(x) + u(x) + u(x - \frac{5x+1}{10}) + \int_{\lambda(x)}^{x} e^{x-s} u(s)ds = f(x), & x \geq 0, \\ u(x) = \frac{5x-x^2}{5}, \quad -\frac{1}{15} \leq x < 0. \end{cases}$$

where $u(x) = xe^{\frac{-x}{5}}$ is exact solution and

$$f(x) = \begin{cases} \frac{1}{180} e^{-\frac{x}{5}} \left(-6x + 125e^{\frac{6x}{5}} + 55\right), & 0 \leq x < \frac{1}{5}, \\ \frac{1}{180} e^{-\frac{x}{5}} \left(-6x + 18e^{\frac{1}{50}(5x+1)}(5x-1) + 5e^{\frac{3x}{5}+\frac{3}{25}}(15x+22) + 55\right), & \frac{1}{5} \leq x. \end{cases}$$

Remark 3.1. There is a very important point in solving problems with the presented RKM in intervals larger than the interval $[0, 1]$. The point is that if the inner product (1.2) and reproducing kernel (1.4) are used, then the approximate solution must be calculated in the desired interval with a much larger number of collocation points (n) compared to the interval $[0, 1]$, because in this case an appropriate approximate solution will be obtained. By comparing Tables 2, 3 with 4, 5 and Figure 3, 4 this point is clearly visible.

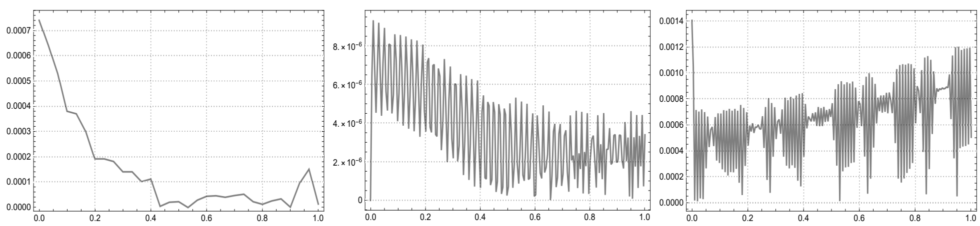

Figure 1: Absolute errors for Example 3.1 (Left: [11], E_{200}; Middle: PM, E_{100}; Right: PM, E'_{100}).

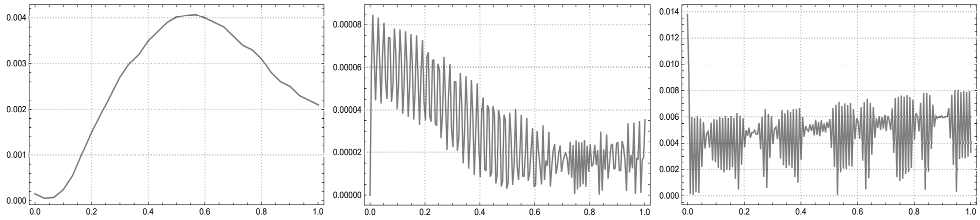

Figure 2: Absolute errors for Example 3.2 (Left: [11], E_{200}; Middle: PM, E_{100}; Right: PM, E'_{100}).

Table 3: Convergence order for Example 3.2 on $x \in [0, 1]$

E_{10}	E_{20}	$Log_2 \frac{E_{10}}{E_{20}}$	E_{40}	$Log_2 \frac{E_{20}}{E_{40}}$	E_{80}	$Log_2 \frac{E_{40}}{E_{80}}$
1.20×10^{-2}	2.80×10^{-3}	2.09954	7.50×10^{-4}	1.90046	1.90×10^{-4}	1.98089
E'_{10}	E'_{20}	$Log_2 \frac{E'_{10}}{E'_{20}}$	E'_{40}	$Log_2 \frac{E'_{20}}{E'_{40}}$	E'_{80}	$Log_2 \frac{E'_{40}}{E'_{80}}$
1.40×10^{-1}	7.00×10^{-2}	1.00	3.50×10^{-2}	1.00	1.70×10^{-2}	1.04182

Table 4: Convergence order for Example 3.1 on $x \in [0, 10]$

E_{40}	E_{80}	$Log_2 \frac{E_{40}}{E_{80}}$	E_{160}	$Log_2 \frac{E_{80}}{E_{160}}$	E_{320}	$Log_2 \frac{E_{160}}{E_{320}}$
3.00×10^{-3}	8.50×10^{-4}	1.81943	2.20×10^{-4}	1.94996	5.60×10^{-5}	1.974
E'_{40}	E'_{80}	$Log_2 \frac{E'_{40}}{E'_{80}}$	E'_{160}	$Log_2 \frac{E'_{80}}{E'_{160}}$	E'_{320}	$Log_2 \frac{E'_{160}}{E'_{320}}$
1.80×10^{-2}	1.00×10^{-2}	0.847997	5.20×10^{-3}	0.943416	2.65×10^{-3}	0.972519

Table 5: Convergence order for Example 3.2 on $x \in [0, 10]$

E_{100}	E_{200}	$Log_2 \frac{E_{100}}{E_{200}}$	E_{400}	$Log_2 \frac{E_{200}}{E_{400}}$	E_{800}	$Log_2 \frac{E_{400}}{E_{800}}$
1.40×10^{-2}	3.30×10^{-3}	2.08489	8.00×10^{-4}	2.04439	1.90×10^{-4}	2.074
E'_{100}	E'_{200}	$Log_2 \frac{E'_{100}}{E'_{200}}$	E'_{400}	$Log_2 \frac{E'_{200}}{E'_{400}}$	E'_{800}	$Log_2 \frac{E'_{400}}{E'_{800}}$
1.90×10^{-1}	1.00×10^{-1}	0.925999	5.30×10^{-2}	0.915936	2.80×10^{-2}	0.920566

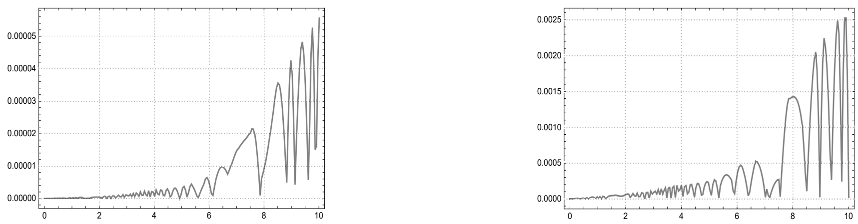

Figure 3: Absolute errors for Example 3.1 (Left: PM, E_{320}; Right: PM, E'_{320}).

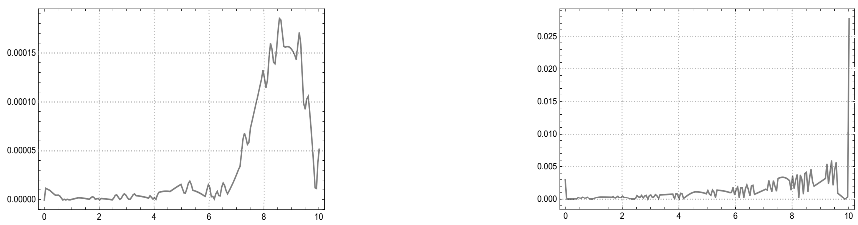

Figure 4: Absolute errors for Example 3.2 (Left: PM, E_{800}; Right: PM, E'_{800}).

4 Conclusion

In this paper, we solved the VDIDE by the different implementation of the RKM, such that makes it unnecessary to use the Gram-Schmidt orthogonalization process. According to Remark 3.1, to achieve an appropriate approximation of the solution in interval $[0, 10]$ by inner product (1.2) and reproducing kernel (1.4), we need a much larger number of collocation points (n). By comparing Tables 2, 3 with 4, 5 it is clear that number of collocation points for Examples 3.1, 3.2 on interval $[0, 1]$ is $n = 10, 20, 40, 80$ and on interval $[0, 10]$ are $n = 40, 80, 160, 360$ and $n = 100, 200, 400, 800$ respectively.

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